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**MA204 - Linear Algebra and Matrices**
**Problem Sheet - 2**

**Matrix Multiplication, Triangular Factorization, Inverse and Transpose**

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1. Working a column at a time of the matrix $A$, compute the product $A\mathbf{x}$, where $A = \begin{bmatrix} 4 & 3 \\ 6 & 6 \\ 8 & 9 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \end{bmatrix}$.

2. If an $m$ by $n$ matrix $A$ multiplies an $n$-dimensional vector $\mathbf{x}$, how many separate multiplications are involved? What if A multiplies an $n$ by $p$ matrix $B$?

3. The first row of $AB$ is a linear combination of all the rows of $B$. What are the coefficients in this combination, and what is the first row of $AB$, if $A = \begin{bmatrix} 2 & 1 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$.

4. Find the condition(s) on $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ so that it commutes with every $2 \times 2$ matrices.

5. Write down the $3 \times 3$ matrices that produce these elimination steps: (a) $E_{21}$ subtracts 5 times row 1 from row 2. (b) $E_{32}$ subtracts $-7$ times row 2 from row 3. (c) $P$ exchanges rows 1 and 2, then rows 2 and 3. Then find $E_{32}E_{21}\mathbf{b}$ and $E_{21}E_{32}\mathbf{b}$. When When $E_{32}$ comes first, row _____ feels no effect from row _____.

6. Suppose $a_{33} = 7$ and the third pivot is 5. If you change $a_{33}$ to 11, the third pivot is _____. If you change $a_{33}$ to _____, there is *zero* in the pivot position.

7. If all columns of $B$ are the same, hen all columns of $AB$ are the same. What about the rows of $AB$, if all rows of $B$ are same? Prove or disprove. Block multiplication separates matrices into blocks (submatrices). If their shapes make block multiplication possible, then it is allowed. Verify following two block matrix multiplications. (Choose appropriate matrices in first case & replace those $x$'s by numbers (distinct) in second case.)

$$\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} AC + BD \end{bmatrix} \qquad \left[\begin{array}{cc|c} x & x & x \\ x & x & x \\ \hline x & x & x \end{array}\right] \left[\begin{array}{cc|c} x & x & x \\ x & x & x \\ \hline x & x & x \end{array}\right] = ?$$

8. True or false? Explain.
   (a) If rows 1 and 3 of $A$ are the same, so are rows 1 and 3 of $AB$.
   (b) $(AB)^2 = A^2B^2$.
   (c) If $A^2$ is defined then $A$ is necessarily square.

(d) If $AB = B$ then $A = I$.

(e) If $A^T$ is invertible, then $A$ is invertible.

9. Factor $A$ into $LU$ and then solve the system $\begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$.

10. Under what conditions is the following product nonsingular?

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

11. How could you factor $A$ into a product $UL$, upper triangular times lower triangular? Would they be the same factors as in $A = LU$?

12. Write down all six of the $3 \times 3$ permutation matrices, including $P = I$. Identify their inverses, which are also permutation matrices. These inverses satisfy $PP^{-1} = I$ and are on the same list.

13. Find the $PA = LDU$ factorizations for $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix}$.

14. Consider $A = \begin{bmatrix} 1 & c & 0 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{bmatrix}$. Which number $c$ leads to zero in the second pivot position? A row exchange is needed and $A = LU$ is not possible. Which $c$ produces zero in the third pivot position? Then a row exchange can't help and elimination fails.

15. If $A = \begin{bmatrix} x & x & x & x \\ x & x & x & 0 \\ 0 & x & x & x \\ 0 & 0 & x & x \end{bmatrix}$ has nonzeros in the positions marked by $x$, which zeros are still zero in their factors $L$ and $U$?

16. How many exchanges will permute $(5,4,3,2,1)$ back to $(1,2,3,4,5)$? How many exchanges to change $(6,5,4,3,2,1)$ to $(1,2,3,4,5,6)$? One is even and the other is odd. For $(n,\ldots,1)$ to $(1,\ldots,n)$, show that $n = 100$ and 101 are even, $n = 102$ and 103 are odd.

17. If $P_1$ and $P_2$are permutation matrices, so is $P_1P_2$. This still has the rows of $I$ in some order. Give examples with $P_1P_2 \neq P_2P_1$ and $P_3P_4 = P_4P_3$.

18. If you take powers of a permutation, why is some $P^k$ eventually equal to $I$? Find a $5 \times 5$ permutation $P$ so that the smallest power to equal $I$ is $P^6$. (This is a challenge question. Combine a 2 by 2 block with a 3 by 3 block.)

19. If $P$ is any permutation matrix, find a nonzero vector $x$ so that $(I - P)x = 0$. (This will mean that $I - P$ has no inverse, and has determinant zero.)

20. From $AB = C$, find a formula for $A^{-1}$. Also find $A^{-1}$ from $PA = LU$.

21. Use the Gauss-Jordan method to invert $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$.

22. Suppose elimination fails because there is no pivot in column 3: $A = \begin{bmatrix} 2 & 1 & 4 & 6 \\ 0 & 3 & 8 & 5 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 9 \end{bmatrix}$. Show that $A$ cannot be invertible. The third row of $A^{-1}$, multiplying $A$, should give the third row $\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$ of $I$. Why is this impossible?

23. If $B$ is square, show that $A = B + B^T$ is always symmetric and $K = B - B^T$ is always skew-symmetric-which means that $K^T = -K$.

24. If $A = L_1 D_1 U_1$ and $A = L_2 D_2 U_2$, ,prove that $L_1 = L_2, D_1 = D_2$, and $U_1 = U_2$. If $A$ is invertible, the factorization is unique.

25. Under what conditions on it's entries is $A = \begin{bmatrix} a & b & c \\ d & e & 0 \\ f & 0 & 0 \end{bmatrix}$ invertible?

26. If A has *column 1* + *column 2* = *column 3*, show that $A$ is not invertible. Find a nonzero solution $\mathbf{x}$ to $A\mathbf{x} = 0$, if $A$ is a $3 \times 3$ matrix.

27. Show that $A^2 = 0$ is possible but $A^T A = 0$ is not possible (unless $A$ = zero matrix).

28. If $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, then find $M^T$. What conditions on $A, B, C$ and $D$ make $M$ symmetric?

29. Prove that no reordering of rows and reordering of columns can transpose a typical matrix.

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